

IDEAL THEORY AND PRÜFER DOMAINS

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OVERRINGS OF ONE-DIMENSIONAL NOETHERIAN RINGS

The main purpose of this lecture is to prove that every overring of a one-dimensional Noetherian domain is again one-dimensional and Noetherian. Throughout this lecture, we let R stands for a commutative ring with identity.

Modules Over Noetherian Rings. In this subsection, we will establish two results related to the set of annihilator of a finitely generated module over a Noetherian ring. Let M be an R -module. Recall that, for each $m \in M$, the set

$$\text{Ann}(m) = \{r \in R : rm = 0\}$$

is called the annihilator of m and $\text{Ann}(M) = \{r \in R : rM = 0\}$ is called the annihilator of M . In this subsection, we let $Z(M)$ denote the set $\bigcup_{m \in M \setminus \{0\}} \text{Ann}(m)$. We need the following lemma.

Lemma 1. *Let M be a nonzero R -module. Then the following statements hold.*

- (1) *If P is maximal in the set $\{\text{Ann}(m) : m \in M \setminus \{0\}\}$, then P is prime.*
- (2) *Every prime ideal minimal over $\text{Ann}(M)$ belongs to $Z(M)$.*

Proof. (1) Let $P = \text{Ann}(m)$ be maximal in the set $\mathcal{A} = \{\text{Ann}(m) : m \in M \setminus \{0\}\}$, and take $r, s \in R$ such that $rs \in P$. Then $r(sm) = 0$. If $sm = 0$, then $s \in P$. Otherwise, $\text{Ann}(sm)$ is an ideal in \mathcal{A} containing r . Since $P \subseteq \text{Ann}(sm)$, the maximality of P ensures that $r \in \text{Ann}(sm) = P$. Thus, the ideal P is prime.

(2) Let P be a prime ideal minimal over $\text{Ann}(M)$. Consider the multiplicative subset $S := \{rt : r \notin Z(M) \text{ and } t \notin P\}$. Note that $R \setminus S \subseteq Z(M)$. Suppose, by way of contradiction, that $S \cap \text{Ann}(M)$ is nonempty and take $rt \in \text{Ann}(M) \cap S$ for some $r \in R \setminus Z(M)$ and $t \in R \setminus P$. Then $rtM = 0$. As $r \notin Z(M)$, the equality $tM = 0$ and so $t \in \text{Ann}(M) \subseteq P$, which is a contradiction. Thus, S is disjoint from $\text{Ann}(M)$. Let Q be maximal among all the ideals containing $\text{Ann}(M)$ and disjoint from S . Then $Q \subseteq Z(M)$ and $\text{Ann}(M) \subseteq Q \subseteq P$. It follows now from the minimality of P that $Q = P$ and, therefore, we can conclude that $P \subseteq Z(M)$. \square

As we proceed to argue, in a zero-dimensional ring, every element that is not a zero-divisor must be a unit.

Proposition 2. *Let R be a zero-dimensional commutative ring with identity. If r is not a zero-divisor, then $r \in R^\times$.*

Proof. Suppose that r is not a zero-divisor. Considering R as a module over itself, we see that $r \notin Z(R)$ and $\text{Ann}(R) = 0$. Since R is zero-dimensional, every maximal ideal of R must be a minimal prime ideal over $\text{Ann}(R)$ and so must be included in $Z(R)$ by part (2) of Lemma 1. Hence r is not contained in any maximal ideal and, therefore, r must be a unit. \square

Proposition 3. *Let R be a Noetherian ring and let M be a finitely generated R -module. Then there are only finitely many ideals of R that are maximal in $Z(M)$. Moreover, each of these ideals is a prime ideal of the form $\text{Ann}(m)$ for some nonzero $m \in M$.*

Proof. Since R is Noetherian and M is finitely generated, M is a Noetherian R -module. Let \mathcal{M} be the set of maximal elements in $\{\text{Ann}(m) : m \in M \setminus \{0\}\}$. We know that every ideal in \mathcal{M} is a prime ideal, and it is clear that the set \mathcal{Z} of elements of R annihilating some nonzero element of M is the union of the prime ideals in \mathcal{M} . Write $\mathcal{M} = \{\text{Ann}(m) : m \in S\}$ for a subset S of M , and let us verify that \mathcal{M} is finite. Let N denote the R -submodule of M spanned by S . As M is Noetherian, N is finitely generated. Write $N = Rm_1 + \cdots + Rm_n$ for some $m_1, \dots, m_n \in S$. Then for any $m \in S$ we can write $m = r_1m_1 + \cdots + r_nm_n$, from which we obtain the inclusion $\text{Ann}(m_1) \cap \cdots \cap \text{Ann}(m_n) \subseteq \text{Ann}(m)$. Because $\text{Ann}(m)$ is prime, $\text{Ann}(m_i) \subseteq \text{Ann}(m)$ for some $i \in \llbracket 1, n \rrbracket$. Now the maximality of $\text{Ann}(m_i)$ implies that $\text{Ann}(m) = \text{Ann}(m_i)$. As a consequence, \mathcal{M} is finite. Finally, suppose that I is an ideal contained in $Z(M)$. Then $I \subseteq \bigcup_{i=1}^n \text{Ann}(m_i)$, and the fact that each $\text{Ann}(m_i)$ is prime implies that $I \subseteq \text{Ann}(m_j)$ for some $j \in \llbracket 1, n \rrbracket$. \square

Proposition 4. *Let R be a Noetherian ring, and let M be a finitely generated nonzero R -module. If P is a prime ideal of R minimal over $\text{Ann}(M)$, then $P = \text{Ann}(m)$ for some $m \in M$.*

Proof. Set $A = \text{Ann}(M)$. It is clear that the R_P -module M_P is a finitely generated module over the Noetherian ring R_P . Let us argue that M_P is nonzero. After writing $M = Rm_1 + \cdots + Rm_k$ for nonzero elements $m_1, \dots, m_k \in M$, we can see that $\bigcap_{i=1}^k \text{Ann}(m_i) \subseteq A \subseteq P$. This, together with the fact that P is prime, allows us to assume that $\text{Ann}(m) \subseteq P$ for a nonzero $m \in M$. Suppose towards a contradiction that $m/1 = 0/1$ in M_P . Then there must be an element $s \in R \setminus P$ with $sm = 0$. Therefore $s \in \text{Ann}(m) \subseteq P$, a contradiction. Thus, $m/1$ is nonzero in M_P , and so M_P is a nonzero R_P -module.

It is clear that A_P is contained in the annihilator of M_P ; indeed, it equals the annihilator of M_P (see Exercise 1), but we do not use this fact in this proof. Let us verify that P_P is minimal over A_P . A prime ideal of R_P between A_P and P_P must have the form Q_P , where Q is a prime ideal of R such that $Q \subseteq P$. Observe that

$A \subseteq {}^c(A_P) \subseteq {}^c(Q_P) = Q$ and $Q = {}^c(Q_P) = {}^c(P_P) = P$, where cJ denotes the contraction of an ideal J of R_P under the localization homomorphism $R \rightarrow R_P$. Since $A \subseteq Q \subseteq P$, the minimality of P ensures that $Q = P$, that is, $Q_P = P_P$. Thus, P_P is minimal over A_P .

It follows now from part (2) of Lemma 1 that P_P is contained in $Z(M_P)$. Since P_P is the maximal ideal of R_P , we see that P_P is, in particular, maximal in the set $Z(M_P)$. As a consequence, Proposition 3 guarantees the existence of an element in M_P whose annihilator is P_P , and we can readily verify that such an element can be taken to be $m/1$ for some $m \in M$. Writing $P = Ra_1 + \cdots + Ra_n$ for some $a_1, \dots, a_n \in R$, we see that P_P is generated by the set $\{a_i/1 : i \in \llbracket 1, n \rrbracket\}$. As $P_P = \text{Ann}(m/1)$, for every $i \in \llbracket 1, n \rrbracket$ there is an $s_i \in R \setminus P$ such that $s_i a_i m = 0$. Then for $s = s_1 \cdots s_n$, the equality $sPm = 0$ holds. Finally, we claim that $P = \text{Ann}(sm)$. It is clear that P annihilates sm . Conversely, if $r \in R$ annihilates sm , then $rs/1$ annihilates $m/1$, and so $rs/1 \in P_P$, that is, $r \in P$. Hence $P = \text{Ann}(sm)$, which concludes the proof. \square

Overrings of One-dimensional Noetherian Domains. In order to prove Theorem 9, we need to introduce the notion of length for modules. Let M be an R -module. A *composition series* of M is a chain

$$(0.1) \quad M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_\ell = 0,$$

where M_j/M_{j+1} is simple, that is, M_j/M_{j+1} has no nonzero proper R -submodule for any $j \in \llbracket 0, \ell - 1 \rrbracket$. In this case, we say that the composition series (0.1) has *length* ℓ . The Jordan-Hölder Theorem states that if M has a composition series, then any chain of R -submodules can be refined to obtain a composition series of M , and that any two composition series of M have the same length. If M has a composition series like (0.1), then ℓ is called the *length* of M .

Recall that the Jacobson radical of R is the intersection of all maximal ideals of R . The following lemma will be used in the proof of Theorem 6.

Lemma 5. *Let R be a zero-dimensional Noetherian ring with identity. Then R has finitely many prime ideals, and $\text{Rad}(0)$ is the Jacobson radical of R .*

Proof. Since R is Noetherian, we know from previous lectures that $\text{Rad}(0)$ is the intersection of finitely many prime ideals, namely, P_1, \dots, P_k (assume they are different). Since every prime ideal P of R contains $\text{Rad}(0)$, we see $P_1 \cdots P_k \subseteq P$. As P is prime, $P_j \subseteq P$ for some $j \in \llbracket 1, k \rrbracket$, and the fact that R is zero-dimension ensures that $P = P_j$. Therefore R has only finitely many prime ideals, which are also maximal ideals. Thus, $\text{Rad}(0)$ is the Jacobson radical of R . \square

We are in a position to characterize zero-dimensional Noetherian rings in terms of composition series.

Theorem 6. *For a commutative ring R with identity, the following statements are equivalent.*

- (a) *R is Noetherian and zero-dimensional.*
- (b) *Every finitely generated R -module has a composition series.*
- (c) *As an R -module, R has a composition series.*

Proof. (a) \Rightarrow (b): Let M be a finitely generated R -module. Since R is a Noetherian zero-dimensional ring, Lemma 5 guarantees that R has finitely many prime ideals, namely, P_1, \dots, P_k . Let $J := \text{Rad}(0)$ be the Jacobson radical of R . As R is Noetherian, J is finitely generated and so nilpotent. Thus, $(P_1 \cdots P_k)^m = (0)$ for some $m \in \mathbb{N}$. Consider the ideals I_1, \dots, I_{km} of R defined by $I_{qm+r} = (P_1 \dots P_q)^m P_{q+1}^r$, where $q \in \llbracket 0, k-1 \rrbracket$ and $r \in \llbracket 1, m \rrbracket$. It is clear that $M \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_{km} = 0$, where $M_j := I_j M$. Now fix $j \in \llbracket 1, km-1 \rrbracket$. Since M_j/M_{j+1} is a finitely generated module over R/P for some $P \in \text{Spec}(R)$ (i.e., M_j/M_{j+1} is a finite-dimensional vector space over the field R/P), there are (R/P) -submodules $M_{j,1}, \dots, M_{j,n_j}$ of M_j containing M_{j+1} such that $M_j = M_{j,1} \supsetneq M_{j,2} \supsetneq \dots \supsetneq M_{j,n_j} = M_{j+1}$ satisfying that, for any $i \in \llbracket 1, n_j-1 \rrbracket$, the quotient $M_{j,i}/M_{j,i+1}$ contains no nontrivial proper (R/P) -submodule and so no nontrivial proper R -submodule. Hence M has a composition series.

(b) \Rightarrow (c): This is clear.

(c) \Rightarrow (a): Since R has a composition series, it has finite length ℓ . Now if $(J_n)_{n \in \mathbb{N}}$ were an ascending chain of ideals (where $J_n \subsetneq J_{n+1}$), then Jordan-Hölder Theorem would allow us to refine the chain $R \supsetneq J_{\ell+1} \supsetneq J_\ell \supsetneq \dots \supsetneq J_1 \supseteq (0)$ to obtain a composition series of R with length at least $\ell + 1$. Hence every ascending chain of ideals of R eventually stabilizes, and so R is Noetherian.

Let us finally argue that R is zero-dimensional. Let P be a prime ideal of R . Since R has a composition series, the integral domain R/P has a composition series as an R/P -module. Thus, it suffices to argue that every integral domain D with a composition series is a field. To prove this, let d be a nonzero element in D . Since D has a composition series, it must have a minimal nonzero ideal I . As $dI \subseteq I$, the minimality of I ensures that $dI = I$. Therefore $d \in I$, and so $d = da$ for some $a \in I$, which implies that $a = 1$. Hence $I = D$, and we can conclude that D is a field. \square

Theorem 6 can be used to prove the following result.

Proposition 7. *Let R be an integral domain. Then R is Noetherian with dimension at most 1 if and only if the R -module R/I has a composition series for every nonzero ideal I .*

Proof. For the direct implication, assume that R is Noetherian with $\dim R \leq 1$, and let I be a nonzero ideal of R . If $\dim R = 0$, then R is a field, and R/I is the zero R -module, which trivially has a composition series. Therefore we suppose that $\dim R = 1$. Observe that the ring R/I is zero-dimensional: indeed, if P is a minimal prime ideal

over I , then the fact that $\dim R = 1$ ensures that P is maximal. Then R/I has a composition series as an (R/I) -module by Theorem 6, and so it has a composition series as an R -module.

For the reverse implication, assume that R/I has a composition series for every nonzero ideal I . If P is a nonzero prime ideal, then R/P has a composition series, and so Theorem 6 guarantees that the integral domain R/P is a zero-dimensional and so a field, whence P is maximal. Hence $\dim R = 1$. Finally, let $(I_n)_{n \in \mathbb{N}_0}$ be an ascending chain of ideals of R with $I_0 \neq (0)$. Then $(I_n/I_0)_{n \in \mathbb{N}}$ is an ascending chain of ideals of R/I_0 . It follows now from Theorem 6 that R/I_0 is Noetherian, and so $(I_n/I_0)_{n \in \mathbb{N}}$ eventually stabilizes. Thus, the same holds for $(I_n)_{n \in \mathbb{N}}$. Hence R is Noetherian. \square

In the proof of Theorem 9, we will use the following technical lemma.

Lemma 8. *Let R be a one-dimensional integral domain, and let a and b be nonzero elements of R . If $J = \{x \in R : xa^n \in Rb \text{ for some } n \in \mathbb{N}\}$, then $J + Ra = R$.*

Proof. Exercise. \square

We are in a position to prove that every overring of a one-dimensional Noetherian domain is both one-dimensional and Noetherian.

Theorem 9. *Let R be a one-dimensional Noetherian domain. Then every overring of R that is not a field is a one-dimensional Noetherian domain.*

Proof. Let T be an overring of R that is not a field. Take a nonzero $a \in R$ and, for every $n \in \mathbb{N}$, set $I_n := Ta^n \cap R + Ra$. It is clear that $(I_n)_{n \in \mathbb{N}}$ is a descending chain of ideals of R , each of them containing Ra . Since R is a one-dimensional Noetherian ring, the R -module R/Ra has a composition series by virtue of Proposition 7. Therefore the descending sequence $(I_n/Ra)_{n \in \mathbb{N}}$ of R -submodules of R/Ra must eventually stabilize. Take $N \in \mathbb{N}$ such that $I_n = I_N$ for every $n \geq N$. We will argue that $T \subseteq Ra^{-N} + Ta$. To do so, take $t := b/c \in T$ for some $b, c \in R$, and then set

$$J := \{x \in R : xa^n \in Rb \text{ for some } n \in \mathbb{N}\}.$$

In light of Lemma 8, the equality $R = J + Ra$ holds. So we can write $1 = j + ra$ for some $j \in J$ and $r \in R$. Take $k \in \mathbb{N}$ such that $ja^k \in Rb$. Now we see that $jt = b(ja^k/c)a^{-k} \in Ra^{-k}$. Therefore $t = (j + ra)t = jt + rat \in Ra^{-k} + Ta$. Now take the minimum $m \in \mathbb{N}$ such that $t \in Ra^{-m} + Ta$.

We claim that $m \leq N$. Suppose, by way of contradiction, that $m > N$. Take $r_1 \in R$ and $t_1 \in T$ such that $t = r_1a^{-m} + t_1a$. Then $r_1 = (t - t_1a)a^m \in Ta^m$, and so $r_1 \in Ta^m \cap R \subseteq I_m$. Since $m > N$, it follows that $I_m = I_{m+1}$, whence we can write $r_1 = t_2a^{m+1} + r_2a$. Hence

$$t = \frac{r_1}{a^m} + t_1a = \frac{t_2a^{m+1} + r_2a}{a^m} + t_1a = \frac{r_2}{a^{m-1}} + (t_1 + t_2)a \in Ra^{-(m-1)} + Ta.$$

However, this generates a contradiction with the minimality of m . As a consequence, $m \leq N$, as desired.

Because $t \in Ra^{-m} + Ta \subseteq Ra^{-N} + Ta$, the inclusion $T \subseteq Ra^{-N} + Ta$ holds. Therefore T/aT is a submodule of a cyclic R -module. As a result, T/aT is a finitely generated R -module. As any nonzero ideal of T contains a nonzero element of R , every quotient of T by a nonzero ideal has a composition series. Hence T is a one-dimensional Noetherian domain by Proposition 7. \square

In general, an overring of a Noetherian domain does not have to be Noetherian, as the following example illustrates.

Example 10. Consider the Noetherian domain $\mathbb{Q}[x, y]$ (we will see in future lectures that $\dim \mathbb{Q}[x, y] = 2$). The quotient field of $\mathbb{Q}[x, y]$ is the ring $\mathbb{Q}(x, y)$ consisting of all rational polynomials in two variables. Now consider the ring $T = \mathbb{Q}[x] + y\mathbb{Q}[x]_x[y]$, where $\mathbb{Q}[x]_x$ is the localization of $\mathbb{Q}[x]$ at the multiplicative set $\{x^n : n \in \mathbb{N}_0\}$ (i.e., the ring of Laurent polynomials $\mathbb{Q}[x, x^{-1}]$). It is clear that T is an overring of R . To argue that T is not Noetherian, it suffices to show that the ideal Ty is not finitely generated. Suppose, otherwise, that $Ty = (f_1, \dots, f_n)$. Take $m \in \mathbb{N}_0$ such that $x^m f_i \in \mathbb{Q}[x, y]$ for all $i \in \llbracket 1, n \rrbracket$. Since $y/x^{m+1} \in Ty$, we can take $g_1, \dots, g_n \in T$ such that the equality

$$(0.2) \quad x^{-1}y = g_1 x^m f_1 + \dots + g_n x^m f_n$$

holds. Then we can equate the coefficients of y in both sides of (0.2) to find that

$$x^{-1} = g_1(x, 0) x^m \frac{d}{dy} f_1(x, 0) + \dots + g_n(x, 0) x^m \frac{d}{dy} f_n(x, 0) \in \mathbb{Q}[x]$$

(here $\frac{d}{dy} h(x, y)$ denotes the formal derivative of $h \in \mathbb{Q}(x)[y]$ with respect to y). However, $x^{-1} \in \mathbb{Q}[x]$ is clearly a contradiction. Thus, we conclude that T is not Noetherian.

EXERCISES

Exercise 1. Let R be a commutative ring with identity, and let M be a finitely generated R -module. For a multiplicative subset S of R , prove that

$$S^{-1} \text{Ann}(M) = \text{Ann}(S^{-1}M).$$

Exercise 2. Let R be a one-dimensional integral domain, and let a and b be nonzero elements of R . Show that if $J = \{x \in R : xa^n \in Rb \text{ for some } n \in \mathbb{N}\}$, then $J + Ra = R$.

Exercise 3. Let R be a one-dimensional Noetherian domain, and let T be an overring of R . For a prime ideal P of R , show that there are only finitely many ideals Q of T lying over P , that is, satisfying $Q \cap R = P$.