

# IDEAL THEORY AND PRÜFER DOMAINS

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## OVERRINGS OF ONE-DIMENSIONAL NOETHERIAN RINGS

The main purpose of this lecture is to prove that every overring of a one-dimensional Noetherian domain is again one-dimensional and Noetherian. Throughout this lecture, we let  $R$  stand for a commutative ring with identity.

**Modules Over Noetherian Rings.** In this subsection, we will establish two results related to the set of annihilator of a finitely generated module over a Noetherian ring. Let  $M$  be an  $R$ -module. Recall that, for each  $m \in M$ , the set

$$\text{Ann}(m) = \{r \in R : rm = 0\}$$

is called the annihilator of  $m$  and  $\text{Ann}(M) = \{r \in R : rM = 0\}$  is called the annihilator of  $M$ . In this subsection, we let  $Z(M)$  denote the set  $\bigcup_{m \in M \setminus \{0\}} \text{Ann}(m)$ . We need the following lemma.

**Lemma 1.** *Let  $M$  be a nonzero  $R$ -module. Then the following statements hold.*

- (1) *If  $P$  is maximal in the set  $\{\text{Ann}(m) : m \in M \setminus \{0\}\}$ , then  $P$  is prime.*
- (2) *Every prime ideal minimal over  $\text{Ann}(M)$  belongs to  $Z(M)$ .*

*Proof.* (1) Let  $P = \text{Ann}(m)$  be maximal in the set  $\mathcal{A} = \{\text{Ann}(m) : m \in M \setminus \{0\}\}$ , and take  $r, s \in R$  such that  $rs \in P$ . Then  $r(sm) = 0$ . If  $sm = 0$ , then  $s \in P$ . Otherwise,  $\text{Ann}(sm)$  is an ideal in  $\mathcal{A}$  containing  $r$ . Since  $P \subseteq \text{Ann}(sm)$ , the maximality of  $P$  ensures that  $r \in \text{Ann}(sm) = P$ . Thus, the ideal  $P$  is prime.

(2) Let  $P$  be a prime ideal minimal over  $\text{Ann}(M)$ . Consider the multiplicative subset  $S := \{rt : r \notin Z(M) \text{ and } t \notin P\}$ . Note that  $R \setminus S \subseteq Z(M)$ . Suppose, by way of contradiction, that  $S \cap \text{Ann}(M)$  is nonempty and take  $rt \in \text{Ann}(M) \cap S$  for some  $r \in R \setminus Z(M)$  and  $t \in R \setminus P$ . Then  $rtM = 0$ . As  $r \notin Z(M)$ , the equality  $tM = 0$  and so  $t \in \text{Ann}(M) \subseteq P$ , which is a contradiction. Thus,  $S$  is disjoint from  $\text{Ann}(M)$ . Let  $Q$  be maximal among all the ideals containing  $\text{Ann}(M)$  and disjoint from  $S$ . Then  $Q \subseteq Z(M)$  and  $\text{Ann}(M) \subseteq Q \subseteq P$ . It follows now from the minimality of  $P$  that  $Q = P$  and, therefore, we can conclude that  $P \subseteq Z(M)$ .  $\square$

As we proceed to argue, in a zero-dimensional ring, every element that is not a zero-divisor must be a unit.

**Proposition 2.** *Let  $R$  be a zero-dimensional commutative ring with identity. If  $r$  is not a zero-divisor, then  $r \in R^\times$ .*

*Proof.* Suppose that  $r$  is not a zero-divisor. Considering  $R$  as a module over itself, we see that  $r \notin Z(R)$  and  $\text{Ann}(R) = 0$ . Since  $R$  is zero-dimensional, every maximal ideal of  $R$  must be a minimal prime ideal over  $\text{Ann}(R)$  and so must be included in  $Z(R)$  by part (2) of Lemma 1. Hence  $r$  is not contained in any maximal ideal and, therefore,  $r$  must be a unit.  $\square$

**Proposition 3.** *Let  $R$  be a Noetherian ring and let  $M$  be a finitely generated  $R$ -module. Then there are only finitely many ideals of  $R$  that are maximal in  $Z(M)$ . Moreover, each of these ideals is a prime ideal of the form  $\text{Ann}(m)$  for some nonzero  $m \in M$ .*

*Proof.* Since  $R$  is Noetherian and  $M$  is finitely generated,  $M$  is a Noetherian  $R$ -module. Let  $\mathcal{M}$  be the set of maximal elements in  $\{\text{Ann}(m) : m \in M \setminus \{0\}\}$ . We know that every ideal in  $\mathcal{M}$  is a prime ideal, and it is clear that the set  $\mathcal{Z}$  of elements of  $R$  annihilating some nonzero element of  $M$  is the union of the prime ideals in  $\mathcal{M}$ . Write  $\mathcal{M} = \{\text{Ann}(m) : m \in S\}$  for a subset  $S$  of  $M$ , and let us verify that  $\mathcal{M}$  is finite. Let  $N$  denote the  $R$ -submodule of  $M$  spanned by  $S$ . As  $M$  is Noetherian,  $N$  is finitely generated. Write  $N = Rm_1 + \cdots + Rm_n$  for some  $m_1, \dots, m_n \in S$ . Then for any  $m \in S$  we can write  $m = r_1m_1 + \cdots + r_nm_n$ , from which we obtain the inclusion  $\text{Ann}(m_1) \cap \cdots \cap \text{Ann}(m_n) \subseteq \text{Ann}(m)$ . Because  $\text{Ann}(m)$  is prime,  $\text{Ann}(m_i) \subseteq \text{Ann}(m)$  for some  $i \in \llbracket 1, n \rrbracket$ . Now the maximality of  $\text{Ann}(m_i)$  implies that  $\text{Ann}(m) = \text{Ann}(m_i)$ . As a consequence,  $\mathcal{M}$  is finite. Finally, suppose that  $I$  is an ideal contained in  $Z(M)$ . Then  $I \subseteq \bigcup_{i=1}^n \text{Ann}(m_i)$ , and the fact that each  $\text{Ann}(m_i)$  is prime implies that  $I \subseteq \text{Ann}(m_j)$  for some  $j \in \llbracket 1, n \rrbracket$ .  $\square$

**Proposition 4.** *Let  $R$  be a Noetherian ring, and let  $M$  be a finitely generated nonzero  $R$ -module. If  $P$  is a prime ideal of  $R$  minimal over  $\text{Ann}(M)$ , then  $P = \text{Ann}(m)$  for some  $m \in M$ .*

*Proof.* Set  $A = \text{Ann}(M)$ . It is clear that the  $R_P$ -module  $M_P$  is a finitely generated module over the Noetherian ring  $R_P$ . Let us argue that  $M_P$  is nonzero. After writing  $M = Rm_1 + \cdots + Rm_k$  for nonzero elements  $m_1, \dots, m_k \in M$ , we can see that  $\bigcap_{i=1}^k \text{Ann}(m_k) \subseteq A \subseteq P$ . This, together with the fact that  $P$  is prime, allows us to assume that  $\text{Ann}(m) \subseteq P$  for a nonzero  $m \in M$ . Suppose towards a contradiction that  $m/1 = 0/1$  in  $M_P$ . Then there must be an element  $s \in R \setminus P$  with  $sm = 0$ . Therefore  $s \in \text{Ann}(m) \subseteq P$ , a contradiction. Thus,  $m/1$  is nonzero in  $M_P$ , and so  $M_P$  is a nonzero  $R_P$ -module.

It is clear that  $A_P$  is contained in the annihilator of  $M_P$ ; indeed, it equals the annihilator of  $M_P$  (see Exercise 1), but we do not use this fact in this proof. Let us verify that  $P_P$  is minimal over  $A_P$ . A prime ideal of  $R_P$  between  $A_P$  and  $P_P$  must have the form  $Q_P$ , where  $Q$  is a prime ideal of  $R$  such that  $Q \subseteq P$ . Observe that

$A \subseteq {}^c(A_P) \subseteq {}^c(Q_P) = Q$  and  $Q = {}^c(Q_P) = {}^c(P_P) = P$ , where  ${}^cJ$  denotes the contraction of an ideal  $J$  of  $R_P$  under the localization homomorphism  $R \rightarrow R_P$ . Since  $A \subseteq Q \subseteq P$ , the minimality of  $P$  ensures that  $Q = P$ , that is,  $Q_P = P_P$ . Thus,  $P_P$  is minimal over  $A_P$ .

It follows now from part (2) of Lemma 1 that  $P_P$  is contained in  $Z(M_P)$ . Since  $P_P$  is the maximal ideal of  $R_P$ , we see that  $P_P$  is, in particular, maximal in the set  $Z(M_P)$ . As a consequence, Proposition 3 guarantees the existence of an element in  $M_P$  whose annihilator is  $P_P$ , and we can readily verify that such an element can be taken to be  $m/1$  for some  $m \in M$ . Writing  $P = Ra_1 + \cdots + Ra_n$  for some  $a_1, \dots, a_n \in R$ , we see that  $P_P$  is generated by the set  $\{a_i/1 : i \in \llbracket 1, n \rrbracket\}$ . As  $P_P = \text{Ann}(m/1)$ , for every  $i \in \llbracket 1, n \rrbracket$  there is an  $s_i \in R \setminus P$  such that  $s_i a_i m = 0$ . Then for  $s = s_1 \cdots s_n$ , the equality  $sPm = 0$  holds. Finally, we claim that  $P = \text{Ann}(sm)$ . It is clear that  $P$  annihilates  $sm$ . Conversely, if  $r \in R$  annihilates  $sm$ , then  $rs/1$  annihilates  $m/1$ , and so  $rs/1 \in P_P$ , that is,  $r \in P$ . Hence  $P = \text{Ann}(sm)$ , which concludes the proof.  $\square$

**Overrings of One-dimensional Noetherian Domains.** In order to prove Theorem 9, we need to introduce the notion of length for modules. Let  $M$  be an  $R$ -module. A *composition series* of  $M$  is a chain

$$(0.1) \quad M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_\ell = 0,$$

where  $M_j/M_{j+1}$  is simple, that is,  $M_j/M_{j+1}$  has no nonzero proper  $R$ -submodule for any  $j \in \llbracket 0, \ell - 1 \rrbracket$ . In this case, we say that the composition series (0.1) has *length*  $\ell$ . The Jordan-Hölder Theorem states that if  $M$  has a composition series, then any chain of  $R$ -submodules can be refined to obtain a composition series of  $M$ , and that any two composition series of  $M$  have the same length. If  $M$  has a composition series like (0.1), then  $\ell$  is called the *length* of  $M$ .

Recall that the Jacobson radical of  $R$  is the intersection of all maximal ideals of  $R$ . The following lemma will be used in the proof of Theorem 6.

**Lemma 5.** *Let  $R$  be a zero-dimensional Noetherian ring with identity. Then  $R$  has finitely many prime ideals, and  $\text{Rad}(0)$  is the Jacobson radical of  $R$ .*

*Proof.* Since  $R$  is Noetherian, we know from previous lectures that  $\text{Rad}(0)$  is the intersection of finitely many prime ideals, namely,  $P_1, \dots, P_k$  (assume they are different). Since every prime ideal  $P$  of  $R$  contains  $\text{Rad}(0)$ , we see  $P_1 \cdots P_k \subseteq P$ . As  $P$  is prime,  $P_j \subseteq P$  for some  $j \in \llbracket 1, k \rrbracket$ , and the fact that  $R$  is zero-dimension ensures that  $P = P_j$ . Therefore  $R$  has only finitely many prime ideals, which are also maximal ideals. Thus,  $\text{Rad}(0)$  is the Jacobson radical of  $R$ .  $\square$

We are in a position to characterize zero-dimensional Noetherian rings in terms of composition series.

**Theorem 6.** *For a commutative ring  $R$  with identity, the following statements are equivalent.*

- (a)  *$R$  is Noetherian and zero-dimensional.*
- (b) *Every finitely generated  $R$ -module has a composition series.*
- (c) *As an  $R$ -module,  $R$  has a composition series.*

*Proof.* (a)  $\Rightarrow$  (b): Let  $M$  be a finitely generated  $R$ -module. Since  $R$  is a Noetherian zero-dimensional ring, Lemma 5 guarantees that  $R$  has finitely many prime ideals, namely,  $P_1, \dots, P_k$ . Let  $J := \text{Rad}(0)$  be the Jacobson radical of  $R$ . As  $R$  is Noetherian,  $J$  is finitely generated and so nilpotent. Thus,  $(P_1 \cdots P_k)^m = (0)$  for some  $m \in \mathbb{N}$ . Consider the ideals  $I_1, \dots, I_{km}$  of  $R$  defined by  $I_{qm+r} = (P_1 \cdots P_k)^m P_{q+1}^r$ , where  $q \in \llbracket 0, k-1 \rrbracket$  and  $r \in \llbracket 1, m \rrbracket$ . It is clear that  $M \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_{km} = 0$ , where  $M_j := I_j M$ . Now fix  $j \in \llbracket 1, km-1 \rrbracket$ . Since  $M_j/M_{j+1}$  is a finitely generated module over  $R/P$  for some  $P \in \text{Spec}(R)$  (i.e.,  $M_j/M_{j+1}$  is a finite-dimensional vector space over the field  $R/P$ ), there are  $(R/P)$ -submodules  $M_{j,1}, \dots, M_{j,n_j}$  of  $M_j$  containing  $M_{j+1}$  such that  $M_j = M_{j,1} \supsetneq M_{j,2} \supsetneq \cdots \supsetneq M_{j,n_j} = M_{j+1}$  satisfying that, for any  $i \in \llbracket 1, n_j-1 \rrbracket$ , the quotient  $M_{j,i}/M_{j,i+1}$  contains no nontrivial proper  $(R/P)$ -submodule and so no nontrivial proper  $R$ -submodule. Hence  $M$  has a composition series.

(b)  $\Rightarrow$  (c): This is clear.

(c)  $\Rightarrow$  (a): Since  $R$  has a composition series, it has finite length  $\ell$ . Now if  $(J_n)_{n \in \mathbb{N}}$  were an ascending chain of ideals (where  $J_n \subsetneq J_{n+1}$ ), then Jordan-Hölder Theorem would allow us to refine the chain  $R \supsetneq J_{\ell+1} \supsetneq J_\ell \supsetneq \cdots \supsetneq J_1 \supseteq (0)$  to obtain a composition series of  $R$  with length at least  $\ell + 1$ . Hence every ascending chain of ideals of  $R$  eventually stabilizes, and so  $R$  is Noetherian.

Let us finally argue that  $R$  is zero-dimensional. Let  $P$  be a prime ideal of  $R$ . Since  $R$  has a composition series, the integral domain  $R/P$  has a composition series as an  $R/P$ -module. Thus, it suffices to argue that every integral domain  $D$  with a composition series is a field. To prove this, let  $d$  be a nonzero element in  $D$ . Since  $D$  has a composition series, it must have a minimal nonzero ideal  $I$ . As  $dI \subseteq I$ , the minimality of  $I$  ensures that  $dI = I$ . Therefore  $d \in I$ , and so  $d = da$  for some  $a \in I$ , which implies that  $a = 1$ . Hence  $I = D$ , and we can conclude that  $D$  is a field.  $\square$

Theorem 6 can be used to prove the following result.

**Proposition 7.** *Let  $R$  be an integral domain. Then  $R$  is Noetherian with dimension at most 1 if and only if the  $R$ -module  $R/I$  has a composition series for every nonzero ideal  $I$ .*

*Proof.* For the direct implication, assume that  $R$  is Noetherian with  $\dim R \leq 1$ , and let  $I$  be a nonzero ideal of  $R$ . If  $\dim R = 0$ , then  $R$  is a field, and  $R/I$  is the zero  $R$ -module, which trivially has a composition series. Therefore we suppose that  $\dim R = 1$ . Observe that the ring  $R/I$  is zero-dimensional: indeed, if  $P$  is a minimal prime ideal

over  $I$ , then the fact that  $\dim R = 1$  ensures that  $P$  is maximal. Then  $R/I$  has a composition series as an  $(R/I)$ -module by Theorem 6, and so it has a composition series as an  $R$ -module.

For the reverse implication, assume that  $R/I$  has a composition series for every nonzero ideal  $I$ . If  $P$  is a nonzero prime ideal, then  $R/P$  has a composition series, and so Theorem 6 guarantees that the integral domain  $R/P$  is a zero-dimensional and so a field, whence  $P$  is maximal. Hence  $\dim R = 1$ . Finally, let  $(I_n)_{n \in \mathbb{N}_0}$  be an ascending chain of ideals of  $R$  with  $I_0 \neq (0)$ . Then  $(I_n/I_0)_{n \in \mathbb{N}}$  is an ascending chain of ideals of  $R/I_0$ . It follows now from Theorem 6 that  $R/I_0$  is Noetherian, and so  $(I_n/I_0)_{n \in \mathbb{N}}$  eventually stabilizes. Thus, the same holds for  $(I_n)_{n \in \mathbb{N}}$ . Hence  $R$  is Noetherian.  $\square$

In the proof of Theorem 9, we will use the following technical lemma.

**Lemma 8.** *Let  $R$  be a one-dimensional integral domain, and let  $a$  and  $b$  be nonzero elements of  $R$ . If  $J = \{x \in R : xa^n \in Rb \text{ for some } n \in \mathbb{N}\}$ , then  $J + Ra = R$ .*

*Proof.* Exercise.  $\square$

We are in a position to prove that every overring of a one-dimensional Noetherian domain is both one-dimensional and Noetherian.

**Theorem 9.** *Let  $R$  be a one-dimensional Noetherian domain. Then every overring of  $R$  that is not a field is a one-dimensional Noetherian domain.*

*Proof.* Let  $T$  be an overring of  $R$  that is not a field. Take a nonzero  $a \in R$  and, for every  $n \in \mathbb{N}$ , set  $I_n := Ta^n \cap R + Ra$ . It is clear that  $(I_n)_{n \in \mathbb{N}}$  is a descending chain of ideals of  $R$ , each of them containing  $Ra$ . Since  $R$  is a one-dimensional Noetherian ring, the  $R$ -module  $R/Ra$  has a composition series by virtue of Proposition 7. Therefore the descending sequence  $(I_n/Ra)_{n \in \mathbb{N}}$  of  $R$ -submodules of  $R/Ra$  must eventually stabilize. Take  $N \in \mathbb{N}$  such that  $I_n = I_N$  for every  $n \geq N$ . We will argue that  $T \subseteq Ra^{-N} + Ta$ . To do so, take  $t := b/c \in T$  for some  $b, c \in R$ , and then set

$$J := \{x \in R : xa^n \in Rc \text{ for some } n \in \mathbb{N}\}.$$

In light of Lemma 8, the equality  $R = J + Ra$  holds. So we can write  $1 = j + ra$  for some  $j \in J$  and  $r \in R$ . Take  $k \in \mathbb{N}$  such that  $ja^k \in Rc$ . Now we see that  $jt = b(ja^k/c)a^{-k} \in Ra^{-k}$ . Therefore  $t = (j + ra)t = jt + rat \in Ra^{-k} + Ta$ . Now take the minimum  $m \in \mathbb{N}$  such that  $t \in Ra^{-m} + Ta$ .

We claim that  $m \leq N$ . Suppose, by way of contradiction, that  $m > N$ . Take  $r_1 \in R$  and  $t_1 \in T$  such that  $t = r_1a^{-m} + t_1a$ . Then  $r_1 = (t - t_1a)a^m \in Ta^m$ , and so  $r_1 \in Ta^m \cap R \subseteq I_m$ . Since  $m > N$ , it follows that  $I_m = I_{m+1}$ , whence we can write  $r_1 = t_2a^{m+1} + r_2a$ . Hence

$$t = \frac{r_1}{a^m} + t_1a = \frac{t_2a^{m+1} + r_2a}{a^m} + t_1a = \frac{r_2}{a^{m-1}} + (t_1 + t_2)a \in Ra^{-(m-1)} + Ta.$$

However, this generates a contradiction with the minimality of  $m$ . As a consequence,  $m \leq N$ , as desired.

Because  $t \in Ra^{-m} + Ta \subseteq Ra^{-N} + Ta$ , the inclusion  $T \subseteq Ra^{-N} + Ta$  holds. Therefore  $T/aT$  is a submodule of a cyclic  $R$ -module. As a result,  $T/aT$  is a finitely generated  $R$ -module. As any nonzero ideal of  $T$  contains a nonzero element of  $R$ , every quotient of  $T$  by a nonzero ideal has a composition series. Hence  $T$  is a one-dimensional Noetherian domain by Proposition 7.  $\square$

In general, an overring of a Noetherian domain does not have to be Noetherian, as the following example illustrates.

**Example 10.** Consider the Noetherian domain  $\mathbb{Q}[x, y]$  (we will see in future lectures that  $\dim \mathbb{Q}[x, y] = 2$ ). The quotient field of  $\mathbb{Q}[x, y]$  is the ring  $\mathbb{Q}(x, y)$  consisting of all rational polynomials in two variables. Now consider the ring  $T = \mathbb{Q}[x] + y\mathbb{Q}[x]_x[y]$ , where  $\mathbb{Q}[x]_x$  is the localization of  $\mathbb{Q}[x]$  at the multiplicative set  $\{x^n : n \in \mathbb{N}_0\}$  (i.e., the ring of Laurent polynomials  $\mathbb{Q}[x, x^{-1}]$ ). It is clear that  $T$  is an overring of  $R$ . To argue that  $T$  is not Noetherian, it suffices to show that the ideal  $Ty$  is not finitely generated. Suppose, otherwise, that  $Ty = (f_1, \dots, f_n)$ . Take  $m \in \mathbb{N}_0$  such that  $x^m f_i \in \mathbb{Q}[x, y]$  for all  $i \in \llbracket 1, n \rrbracket$ . Since  $y/x^{m+1} \in Ty$ , we can take  $g_1, \dots, g_n \in T$  such that the equality

$$(0.2) \quad x^{-1}y = g_1 x^m f_1 + \dots + g_n x^m f_n$$

holds. Then we can equate the coefficients of  $y$  in both sides of (0.2) to find that

$$x^{-1} = g_1(x, 0)x^m \frac{d}{dy} f_1(x, 0) + \dots + g_n(x, 0)x^m \frac{d}{dy} f_n(x, 0) \in \mathbb{Q}[x]$$

(here  $\frac{d}{dy} h(x, y)$  denotes the formal derivative of  $h \in \mathbb{Q}(x)[y]$  with respect to  $y$ ). However,  $x^{-1} \in \mathbb{Q}[x]$  is clearly a contradiction. Thus, we conclude that  $T$  is not Noetherian.

## EXERCISES

**Exercise 1.** Let  $R$  be a commutative ring with identity, and let  $M$  be a finitely generated  $R$ -module. For a multiplicative subset  $S$  of  $R$ , prove that

$$S^{-1}\text{Ann}(M) = \text{Ann}(S^{-1}M).$$

**Exercise 2.** Let  $R$  be a one-dimensional integral domain, and let  $a$  and  $b$  be nonzero elements of  $R$ . Show that if  $J = \{x \in R : xa^n \in Rb \text{ for some } n \in \mathbb{N}\}$ , then  $J + Ra = R$ .

**Exercise 3.** Let  $R$  be a one-dimensional Noetherian domain, and let  $T$  be an overring of  $R$ . For a prime ideal  $P$  of  $R$ , show that there are only finitely many ideals  $Q$  of  $T$  lying over  $P$ , that is, satisfying  $Q \cap R = P$ .